Journal of Computational Physics 229 (2010) 6673–6675

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Short note

A compact artificial viscosity equivalent to a tensor viscosity

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ARTICLE INFO

Article history:
Received 9 March 2010
Received in revised form 17 May 2010
Accepted 22 May 2010
Available online 1 June 2010

Keywords:
Artificial viscosity
Tensor viscosity
Staggered Lagrangian

ABSTRACT

We present a compact artificial viscosity for staggered grid Lagrangian hydrodynamics on polygonal cells in two Cartesian dimensions and using a decomposition into triangles we show that this viscosity is equivalent to a tensor viscosity of Campbell and Shashkov for quadrilaterals.

1. Introduction

One distinguishing feature of staggered Lagrangian schemes is the existence of two sets of quantities: nodal variables and cell variables. The nodal variables are identified by a global nodal index set, likewise the cell variables by a global cell index set. The actual logical and functional relations among nodal quantities and cell quantities will depend on and be defined by each particular application. Each node possesses a time-invariant mass m^i, a position vector $\mathbf{X}_i$, a velocity vector $\mathbf{U}_i\left(\frac{d\mathbf{X}_i}{dt}=\mathbf{U}_i\right)$, and a force vector $\mathbf{f}_i$, whereas the cell variables include time-invariant cell masses m_j (with $\sum_i m^i=\sum_j m_j$), volumes V_j (given functions of the nodal positions), densities ρ_j, specific internal energies e_j, and cell thermodynamic pressures p_j. The force at each node is the sum of the force contributions from the cells that have the node as a vertex. Thus, a semi-discrete system of equations for *momentum* is

$$m^i\frac{d\mathbf{U}_i}{dt}=\mathbf{f}_i=\sum_j \mathbf{g}_{ij}, \tag{1}$$

where $\mathbf{g}_{ij}$ is zero if node i is not a vertex of cell j, and otherwise $\mathbf{g}_{ij}$ is the contribution of cell j to the force at node i.

The focus of this note is on these cell force contributions. So, consider a single polygonal cell (in two cartesian dimensions), or a subcell with nodes labelled $i=1,\dots,I$. Each edge of this polygon has an outward pointing normal with magnitude equal to the magnitude of the edge. The force on the edge is the cell pressure p times this normal. Let $\mathbf{n}_i$ be the average of the two normal vectors of the edges that have node i as a common endpoint. The pressure force contribution at node i is then (suppressing the cell index j)

$$\mathbf{g}_i=p\mathbf{n}_i. \tag{2}$$

Note that momentum is conserved, that is, the sum of the force contributions is zero, since

$$\sum_i \mathbf{n}_i=\mathbf{0}. \tag{3}$$

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0021-9991/$ - see front matter
doi:10.1016/j.jcp.2010.05.034

It is well-known that a viscous force must be added to the pressure force, if for no other reason than to increase entropy in shocks.

The need for a viscous force and some possible forms it might take are indicated in [1], and [2] based on a discretization of the divμgrad**U**. In this note we give a compact definition of a possible viscous force and we show that it is exactly the same as the Campbell–Shashkov viscosity.

2. The viscous force

The proposed viscous force at node i that is to be added to $p\mathbf{n}_i$ in (1) is

$$-\frac{\mu}{V}\sum_{k=1}^{l}(\mathbf{n}_i\cdot\mathbf{n}_k)\mathbf{U}_k, \tag{4}$$

where V is the volume of the polygon, and μ is a non-negative scalar factor associated with the polygon having dimension of density times velocity times length. Note that a low order approximation of the gradient of velocity in the polygon is the dyadic matrix

$$\sum_k \mathbf{U}_k\mathbf{n}_k \equiv V\mathrm{GRAD}(\mathbf{U}) \tag{5}$$

so that (4) generalizes the scalar pressure multiplying the normal vector to a matrix operating on the normal.

We require three key properties of the viscous forces. First, momentum should be conserved. Second, the force should vanish if the cell nodal velocities are equal. Third, this force should be *dissipative*. By (3), the first two are clearly satisfied. For this note it suffices to define dissipation as meaning that the viscous forces do not increase the total kinetic energy of the nodes of the whole domain. Since the kinetic energy of a node is determined by

$$m^i\frac{d\mathbf{U}_i}{dt}\cdot\mathbf{U}_i = \sum_i \mathbf{f}_i\cdot\mathbf{U}_i, \tag{6}$$

it is clearly sufficient that for each cell

$$\sum_i\sum_k \mathbf{n}_i\cdot\mathbf{n}_k\mathbf{U}_k\cdot\mathbf{U}_i \geqslant 0. \tag{7}$$

This is the case since

$$0 \leqslant V^2\mathrm{GRAD}(\mathbf{U}):\mathrm{GRAD}(\mathbf{U}) = \sum_i\sum_k \mathbf{n}_i\cdot\mathbf{n}_k\mathbf{U}_k\cdot\mathbf{U}_i. \tag{8}$$

(Note that in a total energy conserving version of staggered grid Lagrangian hydrodynamics necessarily $m_j\frac{de_j}{dt}+\sum_i\mathbf{f}_i\cdot\mathbf{U}_i=0$, so that the cell internal energy is not decreased by the viscous force.)

3. The Campbell–Shashkov subcell tensor viscosity

We will show that for a quadrilateral cell the Campbell–Shashkov viscosity [1] is equivalent to decomposing the cell into triangles, applying (4) to each triangle, and averaging the resulting force contributions at each node. A different derivation using finite elements to approximate divμ grad**U** can be found in [2].

Consider a quadrilateral with nodes 1, 2, 3, 4 are arranged in counter-clockwise order. Consider node 1. Nodes 1, 2, and 4 form a triangle. Adopting the notation in [1], identify point 1 as point p, point 2 as $p+1$, and point 4 as $p-1$ of the triangle. The triangle has volume V_p and assigned coefficient μ_p. The viscous force increments at the three nodes are

$$\delta\mathbf{f_{p-1}} = -.5\frac{\mu_p}{V_p}\sum_{k=p-1}^{p+1}\mathbf{n}_{p-1}\cdot\mathbf{n}_k\mathbf{U}_k \tag{9}$$

$$\delta\mathbf{f_{p+1}} = -.5\frac{\mu_p}{V_p}\sum_{k=p-1}^{p+1}\mathbf{n}_{p+1}\cdot\mathbf{n}_k\mathbf{U}_k \tag{10}$$

$$\delta\mathbf{f_{p}} = -.5\frac{\mu_p}{V_p}\sum_{k=p-1}^{p+1}\mathbf{n}_{p}\cdot\mathbf{n}_k\mathbf{U}_k = -\delta\mathbf{f_{p-1}}-\delta\mathbf{f_{p+1}}. \tag{11}$$

Now, go back to the quadrilateral indexing, that is, set $\delta\mathbf{f_4}=\delta\mathbf{f_{p-1}}$, $\delta\mathbf{f_2}=\delta\mathbf{f_{p+1}}$, $\delta\mathbf{f_1}=\delta\mathbf{f_p}$. Now repeat this for the triangles with apex at 2, then 3, then 4, and add the increments of the quadrilateral nodal forces to get the final values. Note that there will be a different value of μ for each triangle. Note also that implicit in the above (and in [1] and [2]) is the assumption that the quadrilateral is convex so that $V_p > 0$ for all the triangles.

To see that this is the Campbell–Shashkov viscosity, first introduce the edge normals

$$\mathbf{n_{p,p\pm1}}$$

and note that the triangle nodal normals satisfy

$$\mathbf{n_{p-1}} = -.5\mathbf{n_{p,p+1}} \tag{12}$$
$$\mathbf{n_{p+1}} = -.5\mathbf{n_{p,p-1}} \tag{13}$$
$$\mathbf{n_p} = .5(\mathbf{n_{p,p-1}} + \mathbf{n_{p,p+1}}) \tag{14}$$

to get

$$\delta\mathbf{f_{p-1}} = \frac{.5\mu_p}{4V_p}[(\mathbf{U_p} - \mathbf{U_{p-1}})\mathbf{n_{p,p+1}} \cdot \mathbf{n_{p,p+1}} + (\mathbf{U_p} - \mathbf{U_{p+1}})\mathbf{n_{p,p-1}} \cdot \mathbf{n_{p,p+1}}] \tag{15}$$
$$\delta\mathbf{f_{p+1}} = \frac{.5\mu_p}{4V_p}[(\mathbf{U_p} - \mathbf{U_{p-1}})\mathbf{n_{p,p+1}} \cdot \mathbf{n_{p,p-1}} + (\mathbf{U_p} - \mathbf{U_{p+1}})\mathbf{n_{p,p-1}} \cdot \mathbf{n_{p,p-1}}]. \tag{16}$$

Next, introduce the edge tangent vectors pointing from p to $p \pm 1$

$$\mathbf{l_{p,p\pm1}}$$

and eliminate the normals using

$$|\mathbf{n_{p,p\pm1}}| = |\mathbf{l_{p,p\pm1}}| \tag{17}$$
$$\mathbf{n_{p,p+1}} \cdot \mathbf{n_{p,p-1}} = \mathbf{l_{p,p+1}} \cdot \mathbf{l_{p,p-1}}. \tag{18}$$

From [2]

$$|\mathbf{l_{p,p+1}}|^2|\mathbf{l_{p,p-1}}|^2 \sin^2\theta_p = V_p^2 \tag{19}$$

and

$$|\mathbf{l_{p,p+1}}||\mathbf{l_{p,p-1}}|\cos\theta_p = \mathbf{l_{p,p+1}} \cdot \mathbf{l_{p,p-1}}, \tag{20}$$

where θ_p is the angle at the apex p, we obtain

$$\delta\mathbf{f_{p-1}} = \frac{\mu_p V_p}{8|\mathbf{l_{p,p-1}}|\sin^2\theta_p}\left[\frac{\mathbf{U_p} - \mathbf{U_{p-1}}}{|\mathbf{l}_{\mathbf{p},bfp-\mathbf{1}}|} + \cos\theta_p\frac{\mathbf{U_p} - \mathbf{U_{p+1}}}{|\mathbf{l_{p,p+1|}}}\right] \tag{21}$$
$$\delta\mathbf{f_{p+1}} = \frac{\mu_p V_p}{8|\mathbf{l_{p,p+1}}|\sin^2\theta_p}\left[\frac{\mathbf{U_p} - \mathbf{U_{p+1}}}{|\mathbf{l_{p,p+1}}|} + \cos\theta_p\frac{\mathbf{U_p} - \mathbf{U_{p-1}}}{|\mathbf{l_{p,p-1|}}}\right] \tag{22}$$

which, together with (11), is Campbell–Shashkov.

4. Comments

The decomposition of quadrilaterals into triangles is quite natural, and edge velocity differences appear as a consequence of momentum conservation, but what subcell decomposition is appropriate for other polygonal grids is not clear, and the extension to $r - z$ geometry is even less so. Whether or not this viscosity is effective depends entirely on how μ is defined, and that issue is explored thoroughly in [1,2].

Acknowledgements

The author thanks Mikhail (Misha) Shashkov for sharing his insights and expertise on Lagrangian methods.

This work was performed under the auspices of the National Nuclear Security Administration of the US Department of Energy at Los Alamos National Laboratory, under Contract W-7405-ENG-36 and Contract DE-AC52–06NA25396. The author acknowledges the partial support of the DOE Advance Simulation and Computing (ASC) Program and the DOE Office of Science ASCR Program, and the Laboratory Directed Research and Development program (LDRD) at the Los Alamos National Laboratory.

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